

LINEAR CONGRUENCES WITH RATIOS

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ABSTRACT. We use new bounds of double exponential sums with ratios of integers from prescribed intervals to get an asymptotic formula for the number of solutions to congruences

$$\sum_{j=1}^n a_j \frac{x_j}{y_j} \equiv a_0 \pmod{p},$$

with variables from rather general sets.

1. INTRODUCTION

1.1. Motivation. We count the number of solutions to a linear congruence with rational variables with restricted numerators and denominators. This includes solutions with rationals of a bounded height or more generally with a numerators and denominators from a certain large class of sets with a regular boundary. For example, this class of sets includes all convex sets. In some special cases, the corresponding equation over \mathbb{Q} has recently been considered by Blomer and Brüdern [2] and also by Blomer, Brüdern and Salberger [3]. However, in positive characteristic this natural question has never been studied before.

More precisley, for a prime p we consider the equation

$$(1) \quad \sum_{j=1}^n a_j \frac{x_j}{y_j} = a_0,$$

with coefficients $\mathbf{a} = (a_0, a_1, \dots, a_n) \in \mathbb{F}_p^{n+1}$ and variables

$$\mathbf{x} = (x_1, \dots, x_n), \mathbf{y} = (y_1, \dots, y_n) \in \mathbb{F}_p^n,$$

where \mathbb{F}_p denotes the finite fields of p elements.

Given a set $\mathcal{S} \subseteq [0, p-1]^{2n}$, we use $N(\mathbf{a}; \mathcal{S})$ to denote the number of solutions to the equation (1) with variables $(x_1, y_1, \dots, x_n, y_n) \in \mathcal{S}$.

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The equation (1) can be considered over the integers. In particular, Recently Blomer, Brüdern and Salberger [3] have studied it for $n = 3$, $a_0 = 0$ and $a_1 = a_2 = a_3 = 1$. In particular, by [3, Theorem 1], the number of integers solutions with $(\mathbf{x}, \mathbf{y}) \in [-H, H]^6$, to the analogue of (1) with variables over \mathbb{Z} is given by $H^3 Q(H) + O(H^{3-\delta})$, where $Q \in \mathbb{Q}[X]$ is a polynomial of degree 4 and $\delta > 0$ is some absolute constant. Blomer and Brüdern [2] have also suggested an alternative approach which yields a tight upper bound for the same equation but for a slightly different way of ordering and counting solutions. The methods of [2, 3] can probably be extended to arbitrary n (see, for example, the comment in [3, Section 1.3]).

In [16], a different approach has been suggested, which is based on some arguments from [14] and leads to bounds that are weaker by a logarithmic factor than those expected to be produced by the methods of [2, 3], however it seems to be more robust and is able to work in more general situations.

Here we combine some ideas from [14] with several other arguments and apply them to the case of the equation (1) over a finite field.

Throughout the paper, any implied constants in the symbols O , \ll and \gg may depend on the integer parameter $n \geq 1$. We recall that the notations $U = O(V)$, $U \ll V$ and $V \gg U$ are all equivalent to the statement that the inequality $|U| \leq cV$ holds with some constant $c > 0$.

1.2. Solutions in boxes. We fix some intervals

$$(2) \quad \mathcal{I}_j = [A_j + 1, A_j + K_j], \quad \mathcal{J}_j = [B_j + 1, B_j + L_j] \subseteq [0, p - 1],$$

with integers A_j , B_j , K_j and L_j , $j = 1, \dots, n$, and obtain the following asymptotic formula.

Theorem 1. *For $n \geq 3$ and arbitrary intervals (2) for the box $\mathcal{B} = \mathcal{I}_1 \times \mathcal{J}_1 \times \dots \times \mathcal{I}_n \times \mathcal{J}_n$ we have*

$$\left| N(\mathbf{a}; \mathcal{B}) - \frac{1}{p} \prod_{j=1}^n (K_j L_j) \right| \leq \sqrt{K_1 L_1 K_2 L_2} \prod_{j=3}^n (K_j + \sqrt{p L_j}) p^{o(1)}.$$

We now consider the case when \mathcal{B} is a cube with the side length H .

Corollary 2. *For $n \geq 3$ and intervals (2) with $K_j = L_j = H$, $j = 1, \dots, n$, for the cubic box $\mathcal{C} = \mathcal{I}_1 \times \mathcal{J}_1 \times \dots \times \mathcal{I}_n \times \mathcal{J}_n$ we have*

$$\left| N(\mathbf{a}; \mathcal{C}) - \frac{H^{2n}}{p} \right| \leq p^{n/2-1+o(1)} H^{n/2+1}.$$

In particular, the asymptotic formula of Corollary 2 is nontrivial starting from the values of H of order $p^{n/(3n-2)+\delta}$ for any fixed $\delta > 0$ and sufficiently large p . We also record the following result which is convenient for further applications

For a set $\Omega \subseteq [0, 1]^{2n}$ we use $p\Omega$ to denote its blow up by p , that is,

$$p\Omega = \{p\omega : \omega \in \Omega\}.$$

Rounding up and down the sides of $p\Gamma$ for a cubic box

$$(3) \quad \Gamma = [\alpha_1, \alpha_1 + \xi] \times [\beta_1, \beta_1 + \xi] \times \dots \times [\alpha_n, \alpha_n + \xi] \times [\beta_n, \beta_n + \xi] \in [0, 1]^{2n},$$

we derive

Corollary 3. *For $n \geq 3$ and a cubic box (3) with $\xi > 1/p$ we have*

$$|N(\mathbf{a}; p\Gamma) - \xi^{2n} p^{2n-1}| \leq (\xi^{2n-1} p^{2n-2} + \xi^{n/2+1} p^n) p^{o(1)}.$$

1.3. Solutions in well-shaped sets. We combine Corollary 2 with some ideas of Schmidt [13] to get an asymptotic formula for $N(\mathbf{a}; \Omega)$ for a rather general class of sets, which includes all convex sets.

First we need to introduce some definitions. We define the *distance* between a vector $\alpha \in [0, 1]^m$ and a set $\Xi \subseteq [0, 1]^m$ by

$$\text{dist}(\alpha, \Xi) = \inf_{\beta \in \Xi} \|\alpha - \beta\|,$$

where $\|\gamma\|$ denotes the *Euclidean norm* of γ . Given $\varepsilon > 0$ and a set $\Xi \subseteq [0, 1]^m$ we define the sets

$$\Xi_\varepsilon^+ = \{\alpha \in [0, 1]^m \setminus \Xi : \text{dist}(\alpha, \Xi) < \varepsilon\}$$

and

$$\Xi_\varepsilon^- = \{\alpha \in \Xi : \text{dist}(\alpha, [0, 1]^m \setminus \Xi) < \varepsilon\}.$$

We note that in the definition of Ξ_ε^+ we discard the part of the outer ε -neighbourhood that does not belong to $[0, 1]^m$. These parts can also be included in Ξ_ε^+ but this does not affect our argument as we work only with inner ε -neighbourhoods Ξ_ε^- and $([0, 1]^m \setminus \Xi)_\varepsilon^- = \Xi_\varepsilon^+$.

Following [16] (see also [10, 11]), we say that a set Ξ is *well-shaped* if for every $\varepsilon > 0$ the *Lebesgue measures* $\mu(\Xi_\varepsilon^-)$ and $\mu(\Xi_\varepsilon^+)$ exist, for some constant C , and satisfy

$$(4) \quad \mu(\Xi_\varepsilon^\pm) \leq C\varepsilon.$$

As we have mentioned, all convex sets are well-shaped.

Theorem 4. *For $n \geq 3$ and an arbitrary well-shaped set $\Omega \subseteq [0, 1]^{2n}$ of Lebesgue measure $\mu(\Omega)$, we have*

$$|N(\mathbf{a}; p\Omega) - p^{2n-1} \mu(\Omega)| \leq p^{2n-(5n-4)/(3n-2)+o(1)}.$$

2. PRELIMINARIES

2.1. Multiplicative congruences. We recall the following special case of a result of Ayyad, Cochrane and Zheng [1, Theorem 1]

Lemma 5. *Let $\mathcal{I}_j, \mathcal{J}_j$, $j = 1, 2$, be four intervals as of the form (2)*

$$x_1 y_2 \equiv x_2 y_1 \pmod{p}, \quad x_i \in \mathcal{I}_i, y_i \in \mathcal{J}_i, \quad i = 1, 2$$

has $K_1 K_2 L_1 L_2 / p + O(\sqrt{K_1 K_2 L_1 L_2} p^{o(1)})$ solutions.

We also need a version of the result of Cilleruelo and Garaev [5, Theorem 1].

Lemma 6. *For any integers B, L and M with $0 \leq B < B + L < p$ and $0 \leq M < p$, the congruence*

$$(B + y)z \equiv 1 \pmod{p}, \quad B + 1 \leq y \leq B + L, \quad 1 \leq z \leq M$$

has at most $p^{-1/2+o(1)} L^{1/2} M + p^{o(1)}$ solutions.

Proof. As in the proof of [5, Theorem 1] we note that by the Dirichlet principle, for any positive integers $U < p$ and V with $UV \geq p$ one can choose integers u and v with

$$1 \leq u \leq U, \quad |v| = O(V), \quad uB \equiv v \pmod{p}$$

(see also [6, Lemma 3.2] for a more general statement). With this choice of u and v the above congruence can be written as

$$vz + uyz \equiv u \pmod{p}$$

We now take $U = \lceil (p/L)^{1/2} \rceil$ and $V = \lceil (pL)^{1/2} \rceil$ (thus $UV \geq p$).

Since the left hand side is at most $O(MV + LMU) = O((pL)^{1/2} M)$, we see that for every solution (y, z) we have

$$(5) \quad vz + uyz = u + kp$$

with some integer $k = O((pL)^{1/2} M / p) = O(p^{-1/2} L^{1/2} M)$.

We now recall the well-known bound

$$\tau(m) \leq m^{o(1)},$$

on the number of integer positive divisors $\tau(m)$ of an integer $m \neq 0$, see, for example, [8, Theorem 317]. Since by (5) we have the divisibility $z \mid |u + kp|$ and also $0 < |u + kp| = O(p^2)$, we conclude that for each of the $O(p^{-1/2} L^{1/2} M + 1)$ possible values of k , there are at most $p^{o(1)}$ possible values for z , and thus for y . The result now follows. \square

2.2. Exponential sums with ratios. For a prime p , we denote $\mathbf{e}_p(z) = \exp(2\pi iz/p)$. Clearly for $p \nmid u$ the expression $\mathbf{e}_p(av/u)$ is correctly defined (as $\mathbf{e}_p(aw)$ for $w \equiv v/u \pmod{p}$).

Let

$$(6) \quad \mathcal{I} = [A+1, A+K], \quad \mathcal{J} = [B+1, B+L] \subseteq [0, p-1],$$

be two intervals with integers A, B, K and L .

The following result is a variation of [14, Lemma 3]. We present it a slightly more general form that we need for our applications.

Lemma 7. *Let \mathcal{I} and \mathcal{J} be two intervals of the form (6) and let $\mathcal{W} \subseteq \mathcal{I} \times \mathcal{J}$ be an arbitrary convex set. Then uniformly over the integers a with $\gcd(a, p) = 1$, we have*

$$\sum_{(x,y) \in \mathcal{W}} \mathbf{e}_p(ax/y) \ll (K + p^{1/2}L^{1/2})p^{o(1)},$$

where the summation is over all integral points $(x, y) \in \mathcal{W}$.

Proof. Since \mathcal{W} is convex, for each y we there are integers $K \geq K_y > H_y \geq 1$ such that

$$\sum_{(x,y) \in \mathcal{W}} \mathbf{e}_p(ax/y) = \sum_{y \in \mathcal{J}} \sum_{x=A+H_y}^{A+K_y} \mathbf{e}_p(ax/y).$$

Following the proof of [14, Lemma 3], we define

$$I = \lfloor \log(2p/K) \rfloor \quad \text{and} \quad J = \lfloor \log(2p) \rfloor.$$

Furthermore, for a rational number $\alpha = u/v$ with $\gcd(v, p) = 1$, we denote by $\rho(\alpha)$ the unique integer w with $w \equiv u/v \pmod{p}$ and $|w| < p/2$. Using the bound

$$\sum_{x=A+H_y}^{A+K_y} \mathbf{e}_p(\alpha x) \ll \min \left\{ K, \frac{p}{|\rho(\alpha)|} \right\},$$

which holds for any rational α with the denominator that is not a multiple of p (see [9, Bound (8.6)]), we obtain a version of [14, Equation (1)]:

$$(7) \quad \sum_{(x,y) \in \mathcal{W}} \mathbf{e}_p(ax/y) \ll KR + p \sum_{j=I+1}^J T_j e^{-j},$$

where

$$R = \# \{y : B+1 \leq y \leq B+L, |\rho(a/y)| < e^I\},$$

$$T_j = \# \{y : B+1 \leq y \leq B+L, e^j \leq |\rho(a/y)| < e^{j+1}\}.$$

We now see that Lemma 6 implies the bounds

$$R \leq p^{-1/2+o(1)} L^{1/2} e^I + p^{o(1)} \leq p^{1/2+o(1)} L^{1/2} K^{-1} + p^{o(1)}$$

and

$$T_j \leq p^{-1/2+o(1)} L^{1/2} e^j + p^{o(1)}.$$

Substituting these bounds in (7), we obtain

$$\begin{aligned} & \left| \sum_{(x,y) \in \mathcal{W}} \mathbf{e}_p(ax/y) \right| \\ & \ll p^{1/2+o(1)} L^{1/2} + K p^{o(1)} + p \sum_{j=I+1}^J (p^{-1/2+o(1)} L^{1/2} e^j + p^{o(1)}) e^{-j} \\ & = p^{1/2+o(1)} L^{1/2} + K p^{o(1)} + J p^{1/2+o(1)} L^{1/2} + p^{1+o(1)} e^{-I} \\ & = p^{1/2+o(1)} L^{1/2} + K p^{o(1)}, \end{aligned}$$

which concludes the proof. \square

We also need a version of Lemma 7 on average over a .

Lemma 8. *Let \mathcal{I} and \mathcal{J} be two intervals of the form (6). Then, we have*

$$\sum_{a=1}^{p-1} \left| \sum_{x \in \mathcal{I}} \sum_{y \in \mathcal{J}} \mathbf{e}_p(ax/y) \right|^2 \leq K L p^{1+o(1)}.$$

Proof. First we write

$$(8) \quad \sum_{a=1}^{p-1} \left| \sum_{x \in \mathcal{I}} \sum_{y \in \mathcal{J}} \mathbf{e}_p(ax/y) \right|^2 = \sum_{a=0}^{p-1} \left| \sum_{x \in \mathcal{I}} \sum_{y \in \mathcal{J}} \mathbf{e}_p(ax/y) \right|^2 - K^2 L^2.$$

Expanding the square of the inner sum on the right hand side of (8), changing the order of summations and using the orthogonality of characters, we obtain

$$\sum_{a=0}^{p-1} \left| \sum_{x \in \mathcal{I}} \sum_{y \in \mathcal{J}} \mathbf{e}_p(ax/y) \right|^2 = \sum_{x_1, x_2 \in \mathcal{I}} \sum_{y_1, y_2 \in \mathcal{J}} \sum_{a=0}^{p-1} \mathbf{e}_p(a(x_1/y_1 - x_2/y_2)) = pT,$$

where T is the number of solutions to the congruence

$$(9) \quad x_1/y_1 \equiv x_2/y_2 \pmod{p}, \quad x_1, x_2 \in \mathcal{I}, y_1, y_2 \in \mathcal{J}.$$

Extending the admissible region of solutions to $\mathcal{I} \times \mathcal{J}$ and evoking Lemma 5, we conclude that

$$T = \frac{K^2 L^2}{p} + O(KLp^{o(1)})$$

which together with (8) completes the proof. \square

3. PROOFS OF MAIN RESULTS

3.1. Proof of Theorem 1. Using the orthogonality of the exponential function, we write

$$N(\mathbf{a}; \mathcal{B}) = \sum_{(x_1, y_1, \dots, x_n, y_n) \in \mathcal{B}} \dots \sum_{\lambda=0}^{p-1} \frac{1}{p} \sum_{\lambda=0}^{p-1} \mathbf{e}_p \left(\lambda \left(\sum_{j=1}^n a_j \frac{x_j}{y_j} - a_0 \right) \right).$$

Changing the order of summation, and recalling the \mathcal{B} is a direct product of the intervals \mathcal{I}_j and \mathcal{J}_j , $j = 1, \dots, n$, we obtain

$$N(\mathbf{a}; \mathcal{B}) = \frac{1}{p} \sum_{\lambda=0}^{p-1} \mathbf{e}_p(-\lambda a_0) \prod_{j=1}^n \sum_{x_j \in \mathcal{I}_j} \sum_{y_j \in \mathcal{J}_j} \mathbf{e}_p(\lambda a_j x_j / y_j).$$

Now, the contribution from $\lambda = 0$ gives the main term

$$\frac{1}{p} \prod_{j=1}^n \sum_{x_j \in \mathcal{I}_j} \sum_{y_j \in \mathcal{J}_j} 1 = \frac{1}{p} \prod_{j=1}^n (K_j L_j).$$

To estimate the error term, we apply Lemma 7 to $n - 2$ sums with $j = 3, \dots, n$, getting

$$(10) \quad N(\mathbf{a}; \mathcal{B}) - \frac{1}{p} \prod_{j=1}^n (K_j L_j) \leq p^{-1+o(1)} \prod_{j=3}^n (K_j + p^{1/2} L_j^{1/2}) W,$$

where

$$W = \sum_{\lambda=1}^{p-1} \left| \sum_{x_1 \in \mathcal{I}_1} \sum_{y_1 \in \mathcal{J}_1} \mathbf{e}_p(\lambda a_1 x_1 / y_1) \right| \left| \sum_{x_2 \in \mathcal{I}_2} \sum_{y_2 \in \mathcal{J}_2} \mathbf{e}_p(\lambda a_2 x_2 / y_2) \right|.$$

Hence, by the Cauchy inequality,

$$(11) \quad W \leq \sqrt{W_1 W_2},$$

where, for $\nu = 1, 2$,

$$W_\nu = \sum_{\lambda=1}^{p-1} \left| \sum_{x_\nu \in \mathcal{I}_\nu} \sum_{y_\nu \in \mathcal{J}_\nu} \mathbf{e}_p(\lambda a_\nu x_\nu / y_\nu) \right|^2 = \sum_{a=1}^{p-1} \left| \sum_{x_\nu \in \mathcal{I}_\nu} \sum_{y_\nu \in \mathcal{J}_\nu} \mathbf{e}_p(ax_\nu / y_\nu) \right|^2.$$

We now apply Lemma 8 to estimate W_1 and W_2 and see from (11) that

$$W \leq \sqrt{K_1 L_1 K_2 L_2} p^{1+o(1)}$$

which together with (10) concludes the proof.

3.2. Proof of Corollaries 2 and 3. For Corollary 2, we see that the first terms, appearing in the bound of Theorem 1 is H^2 while each term in the product becomes $O(p^{1/2} H^{1/2})$. The result now follows.

For Corollary 3, we approximate the set $p\Gamma$ by two cubes with side lengths $\lfloor \xi p \rfloor$ and $\lceil \xi p \rceil$. Since $\xi > 1/p$, we have $(\xi p + O(1))^{2n} = (\xi p)^{2n} + O((\xi p)^{2n-1})$. The result now follows from Corollary 2.

3.3. Proof of Theorem 4. The proof follows the arguments of the proofs of [11, Theorem 1] or [15, Theorem 3.1] (however the concrete details are different).

First we observe that since the complementary set $[0, 1]^{2n} \setminus \Omega$ is also well-shaped, it is enough to establish only the lower bound

$$(12) \quad N(\mathbf{a}; p\Omega) \geq \frac{N(p\Omega)}{p} + O(p^{2n-4/3+o(1)}).$$

We now recall some constructions and arguments from the proof of [13, Theorem 2]. Pick a point $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{2n}) \in [0, 1]^{2n}$ such that all its coordinates are irrational. For a positive integer k , let $\mathfrak{C}(k)$ be the set of cubes of the form

$$\left[\alpha_1 + \frac{u_1}{k}, \alpha_1 + \frac{u_1 + 1}{k} \right] \times \dots \times \left[\alpha_{2n} + \frac{u_{2n}}{k}, \alpha_{2n} + \frac{u_{2n} + 1}{k} \right],$$

with $u_1, \dots, u_{2n} \in \mathbb{Z}$.

We consider the set of points

$$(13) \quad \left(\frac{x_1}{p}, \frac{y_1}{p}, \dots, \frac{x_n}{p}, \frac{y_n}{p} \right) \in [0, 1]^{2n}$$

taken over all solutions $(\mathbf{x}, \mathbf{y}) \in \mathbb{F}_p^{2n}$ to the equation (1)

Note that the above irrationality condition on $\boldsymbol{\alpha}$ guarantees that the points (13) all belong to the interior of the cubes from $\mathfrak{C}(k)$.

Furthermore, let $\mathfrak{C}_0(k)$ be the set of cubes from $\mathfrak{C}(k)$ that are contained inside of Ω . By [13, Equation (9)], for any well-shaped set $\Omega \in [0, 1]^{2n}$, we have

$$(14) \quad \#\mathfrak{C}_0(k) = k^{2n} \mu(\Omega) + O(k^{2n-1}).$$

Let $\mathfrak{B}_1 = \mathfrak{C}_0(2)$ and for $i = 2, 3, \dots$, let \mathfrak{B}_i be the set of cubes $\Gamma \in \mathfrak{C}_0(2^i)$ that are not contained in any cube from $\mathfrak{C}_0(2^{i-1})$. Clearly

$$(15) \quad 2^{-2in} \# \mathfrak{B}_i + 2^{-2(i-1)n} \# \mathfrak{C}_0(2^{i-1}) \leq \mu(\Omega), \quad i = 2, 3, \dots$$

We now infer from (14) that

$$\begin{aligned} \mu(\Omega) - 2^{-2(i-1)n} \# \mathfrak{C}_0(2^{i-1}) \\ = \mu(\Omega) - 2^{-2(i-1)n} (2^{2(i-1)n} \mu(\Omega) + O(2^{(i-1)(2n-1)})) \\ \ll 2^{(i-1)(2n-1)-2(i-1)n} = 2^{-i+1}. \end{aligned}$$

Therefore, the inequality (15) implies the bound

$$(16) \quad \# \mathfrak{B}_i \ll 2^{i(2n-1)}.$$

We also see that for any integer $M \geq 1$,

$$(17) \quad \Omega \setminus \Omega_\varepsilon^- \subseteq \bigcup_{i=1}^M \bigcup_{\Gamma \in \mathfrak{B}_i} \Gamma \subseteq \Omega$$

with $\varepsilon = (2n)^{1/2} 2^{-M}$. Indeed, for any point $\gamma \in \Omega \setminus \Omega_\varepsilon^-$ there is a cube $\Gamma_\gamma \in \mathfrak{C}(2^M)$ with $\gamma \in \Gamma$ (since for any integer $k \geq 1$, the cubes from $\mathfrak{C}(k)$ tile the whole space \mathbb{R}^{2n}). Because the diameter (that is, the largest distance between the points) of Γ_γ is $(2n)^{1/2} 2^{-M}$, we see from the definition of Ω_ε^- that $\Gamma_\gamma \cap [0, 1]^{2n} \setminus \Omega = \emptyset$. Thus $\Gamma_\gamma \subseteq \Omega$. This implies

$$\Gamma_\gamma \subseteq \bigcup_{i=1}^{2n} \bigcup_{\Gamma \in \mathfrak{B}_i} \Gamma$$

and (17) follows.

Since Ω is well-shaped, from (4) we deduce that

$$(18) \quad \mu \left(\bigcup_{i=1}^{2n} \bigcup_{\Gamma \in \mathfrak{B}_i} \Gamma \right) = \mu(\Omega) + O(2^{-M}).$$

We now assume that

$$(19) \quad 2^M < p$$

so Corollary 3 applies to all cubes $\Gamma \in \mathfrak{C}_0(2^i)$, $i = 1, \dots, M$. Together with (17), this implies the inequality:

$$(20) \quad N(\mathbf{a}; p\Omega) \geq \sum_{i=1}^M \sum_{\Gamma \in \mathfrak{B}_i} N(\mathbf{a}; p\Gamma) = p^{2n-1} \sum_{i=1}^M \sum_{\Gamma \in \mathfrak{B}_i} \mu(\Gamma) + O(R),$$

where

$$R = \sum_{i=1}^M \# \mathfrak{B}_i \left(2^{-i(2n-1)} p^{2n-2} + 2^{-i(n/2+1)} p^n \right) p^{o(1)}.$$

We see from (18) that

$$(21) \quad p^{2n-1} \sum_{i=1}^M \sum_{\Gamma \in \mathfrak{B}_i} \mu(\Gamma) = p^{2n-1} \mu \left(\bigcup_{i=1}^M \bigcup_{\Gamma \in \mathfrak{B}_i} \Gamma \right) \\ = p^{2n-1} \mu(\Omega) + O(p^{2n-1} 2^{-M}).$$

Furthermore, using (16), we derive

$$(22) \quad R \leq \sum_{i=1}^M \left(p^{2n-2} + 2^{i(3n/2-2)} p^n \right) p^{o(1)} \\ = \left(M p^{2n-2} + 2^{M(3n/2-2)} p^n \right) p^{o(1)}.$$

Substituting (21) and (22) in (20) with the above choice of M , noticing that (19) implies $M = O(\log p)$, we obtain

$$(23) \quad N(\mathbf{a}; p\Omega) \geq p^{2n-1} \mu(\Omega) - Q p^{o(1)},$$

where

$$(24) \quad Q \leq p^{2n-1} 2^{-M} + p^{2n-2} + 2^{M(3n/2-2)} p^n.$$

We now choose M to satisfy

$$2^M \leq p^{2(n-1)/(3n-2)} < 2^{M+1},$$

which asymptotically optimises the right hand side of the bound (24), verifies (19) and produces to the bound $Q \ll p^{2n-(5n-4)/(3n-2)} + p^{2n-2} \ll p^{2n-5n/(3n-2)}$. We now see from (23) that (12) holds, which concludes the proof.

4. COMMENTS

We note that for $B_1 = \dots = B_n = 0$, using [14, Lemma 3] instead of Lemma 7 in this special case one can improve Theorem 1 as follows

$$\left| N(\mathbf{a}; \mathcal{B}) - \frac{1}{p} \prod_{j=1}^n (K_j L_j) \right| \\ \leq \left(\frac{K_1 L_1}{p^{1/2}} + \sqrt{K_1 L_1} \right) \left(\frac{K_2 L_2}{p^{1/2}} + \sqrt{K_2 L_2} \right) \prod_{j=3}^n (K_j + L_j) p^{o(1)}.$$

Furthermore, it is easy to see that one can get a version of Lemma 8 for the more general sums of Lemma 7, which becomes

$$\sum_{a=1}^{p-1} \left| \sum_{(x,y) \in \mathcal{W}} \mathbf{e}_p(ax/y) \right|^2 \leq K^2 L^2 + K L p^{1+o(1)},$$

that is, there is no cancellation of the main term for the number of solutions to the congruence (9) anymore. Thus the same arguments lead to the following result. For $n \geq 3$ and arbitrary intervals (2) and arbitrary convex sets $\mathcal{W}_j \subseteq \mathcal{I}_j \times \mathcal{J}_j$, $j = 1, \dots, n$, for the set $\mathcal{S} = \mathcal{W}_1 \times \dots \times \mathcal{W}_n$ we have

$$\begin{aligned} & \left| N(\mathbf{a}; \mathcal{S}) - \frac{N(\mathcal{S})}{p} \right| \\ & \leq \left(\frac{K_1 L_1}{p^{1/2}} + \sqrt{K_1 L_1} \right) \left(\frac{K_2 L_2}{p^{1/2}} + \sqrt{K_2 L_2} \right) \prod_{j=3}^n (K_j + \sqrt{p L_j}) p^{o(1)}, \end{aligned}$$

where $N(\mathcal{S}) = \#(\mathcal{S} \cap \mathbb{Z}^{2n})$. For example, this can be used for counting solutions to the equation (1) with variables in disks

$$(x_j - b_j)^2 + (y_j - c_j)^2 \leq r_j^2, \quad j = 1, \dots, n.$$

One can also ask about solutions to (1) with additional co-primality condition $\gcd(x_j, y_j) = 1$, $j = 1, \dots, n$, that is, essentially in *Farey fractions*. Using simple inclusion-exclusion arguments, one can easily derive relevant asymptotic formulas from our results.

Finally, we remark that Lemma 7 can be viewed as a statement about cancellations among short Kloosterman sums of the form

$$\mathcal{K}(\lambda; \mathcal{J}) = \sum_{u \in \mathcal{J}} \mathbf{e}_p(\lambda/u)$$

over an interval $\mathcal{J} = [B + 1, B + L]$ when λ runs over an interval $\mathcal{I} = [A + 1, A + K]$. Say, for $K = L$ we have a nontrivial cancellation starting with $L \geq p^{1/3+\delta}$ for any fixed $\delta > 0$, which is beyond the range of modern bounds of individual sums short Kloosterman sums over intervals that are not at the origin, we refer to the recent work of Bourgain and Garaev [4] for an outline of the state of art and several results.

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